XXV. *On the fluents of irrational functions.* *By* Edward Ffrench Bromhead, *Esq. M. A.* *Communicated by* J. F. W. Herschel, *Esq. F. R. S.*

Read June 4, 1816.

The efforts of analysts in determining the fluents of rational functions, have been completely successful, and their labours form one of the most perfect and beautiful branches of the fluxionary calculus. In the irrational functions, however, we find but little effected. With the exception of Waring, modern analysts have not added any thing important, to the forms given by Newton, Craig, Cotes, and Bernoulli. No attempt has been made to generalize the known forms, and the last eminent writer on the subject, La Croix, seems to consider them as independent results, not deducible from any common principles, and refers us to the Petersburgh Acts, and other miscellaneous Collections. In the following pages, it is attempted to generalize and systematize our knowledge on this subject; and to show that all the known forms result from other forms of the greatest extent, not depending on particular functions, but upon the properties of all rational functions whatever.

R, R_1, R_2, R_n, denote rational functions of any kind; $R^{-1}, R_1^{-1}, R_2^{-1}, R_n^{-1}$ their inverse functions. Thus if $x = R(v)$ any rational function of (v), then $v = R^{-1}(x)$ the inverse function.

It is thought unnecessary to prove, that the fluxions of all rational functions, and all rational functions of them, are themselves rational.

PROP. I.

$$dx \,.\, R\left\{x, \underset{1}{R}^{-1}(x)\right\} \text{ can be rationalized.}$$

Let $\underset{1}{R}^{-1}(x) = v$; $x = \underset{1}{R}(v)$; $dx = dv \,.\, D\underset{1}{R}(v)$*

which substituted produce the rational form

$$dv \,.\, D\underset{1}{R}(v) \,.\, R\left\{\underset{1}{R}(v), v\right\}$$

Cor. 1. This form includes

$$dx \,.\, R\left\{x, \underset{1}{R}^{-1}(x), \underset{2}{R}\left(x, \underset{1}{R}^{-1}(x)\right), \ldots \underset{n}{R}\left(x, \underset{1}{R}^{-1}(x)\right)\right\}$$

Cor. 2. We may find, a priori, what fluents will come under this form. For let $x = \underset{1}{R}(v)$ any rational function whatever

$$= \frac{\underset{0}{\alpha} + \underset{1}{\alpha} \,.\, v + \underset{2}{\alpha} \,.\, v^2 + \ldots}{\underset{0}{a} + \underset{1}{a} \,.\, v + \underset{2}{a} \,.\, v^2 + \ldots}$$

which is the general form taken by rational functions, when the integer powers are expanded, and the fractions reduced to a common denominator, the coefficients being positive, negative, or nothing. Hence we have $\underset{1}{R}^{-1}(x) = v$, determined from this equation.

$$\left(\underset{0}{a} \,.\, x - \underset{0}{\alpha}\right) + \left(\underset{1}{a} \,.\, x - \underset{1}{\alpha}\right) . v + \left(\underset{2}{a} \,.\, x - \underset{2}{\alpha}\right) . v^2 + \ldots = o$$

Cor. 3. Let the equation be

$$(ax - \alpha) + (bx - \beta)\, v^n = o$$

Then $\underset{1}{R}^{-1}(x) = v = \left\{\frac{-ax + \alpha}{bx - \beta}\right\}^{\frac{1}{n}}$

$\therefore$ we know $\int dx \,.\, R\left\{x, \left(\frac{ax + \alpha}{bx - \beta}\right)^{\frac{1}{n}}\right\}$

* See Note at the end of the Paper.

Cor. 4. Let the equation be

$$(ax - \alpha) + (bx - \beta) \cdot v^n + (cx - \gamma) \cdot v^{2n} = o$$

from which we can determine

$$\int dx \cdot \mathrm{R} \left\{ x, \left(- \frac{bx - \beta}{2cx - 2\gamma} \pm \sqrt{\left(\frac{bx - \beta}{2cx - 2\gamma} \right)^2 - \frac{ax - \alpha}{cx - \gamma}} \right)^{\frac{1}{n}} \right\}$$

and therefore any fluent of the form

$$\int dx \cdot \mathrm{R} \left\{ x, \sqrt[n]{\alpha x + \delta \pm \sqrt{\alpha^2 x^2 + \beta x + \gamma}} \right\}$$

It is obvious that these deductions may be carried to any extent, producing forms hitherto supposed impracticable.

Cor. 5. $\frac{\alpha x + \beta}{ax + b}$ is both of the form $\mathrm{R}^{-1}(x)$ and $\mathrm{R}^{+1}(x)$.

PROP. II.

We can rationalize

$$dx \cdot \mathrm{R} \left\{ x, \underset{1}{\mathrm{R}}^{-1}(x), \underset{2}{\mathrm{R}}^{-1} \underset{1}{\mathrm{R}}^{-1}(x), \ldots \underset{n}{\mathrm{R}}^{-1} \underset{n-1}{\mathrm{R}}^{-1} \ldots \underset{2}{\mathrm{R}}^{-1} \underset{1}{\mathrm{R}}^{-1}(x) \right\}$$

Let $\underset{n}{\mathrm{R}}^{-1} \ldots \underset{1}{\mathrm{R}}^{-1}(x) = v$. Then

$$x = \underset{1}{\mathrm{R}} \, \underset{2}{\mathrm{R}} \ldots \underset{n}{\mathrm{R}} (v)$$

$$\underset{1}{\mathrm{R}}^{-1} x = \underset{2}{\mathrm{R}} \ldots \underset{n}{\mathrm{R}} (v)$$

&c. = &c.

which substituted in the original expression, make it rational.

Cor. 1. If $\underset{1}{\mathrm{R}} = \underset{2}{\mathrm{R}} = \underset{n}{\mathrm{R}}$, the fluxion becomes

$$dx \cdot \mathrm{R} \left\{ x, \underset{1}{\mathrm{R}}^{-1}(x), \underset{1}{\mathrm{R}}^{-2}(x), \ldots \underset{1}{\mathrm{R}}^{-n}(x) \right\}$$

Cor. 2. By this theorem, any of the expressions deducible from Prop. I. may enter contemporaneously, and we may find fluents of very great intricacy.

Cor 3. The fluents

$$\int dx \cdot \mathrm{R} \left\{ x, (\alpha + \beta x)^{\frac{1}{n}}, \left(\frac{a + b \sqrt[n]{\alpha + \beta x}}{c + d \sqrt[n]{\alpha + \beta x}} \right)^{\frac{1}{m}}, \ldots \right\}$$

$$\int dx \,.\, \mathrm{R}\left\{x, \sqrt[n]{a+\sqrt[m]{b+\sqrt[r]{\ldots\sqrt[p]{\mathrm{R}^{-1}(x),\ldots}}}}\right\}$$

and some of the most complex expressions in WARING's Med. Anal. are very particular cases of this form.

PROP. III.

We can rationalize

$$dx \,.\, \mathrm{R}\left\{x, \underset{1}{\mathrm{R}}^{-1}(x), \left(\underset{1}{\mathrm{R}}^{-1}(x)\right)^{\frac{1}{m}}, \left(\underset{1}{\mathrm{R}}^{-1}(x)\right)^{\frac{1}{n}}, \left(\underset{1}{\mathrm{R}}^{-1}(x)\right)^{\frac{1}{r}}, \ldots\right\}$$

$$\text{Let } \underset{1}{\mathrm{R}}^{-1}(x) = v^{m\,.\,n\,.\,r\,.\,.}$$

$$\text{Then } x = \underset{1}{\mathrm{R}}\left(v^{m\,.\,n\,.\,r\,.\,.\,.}\right)$$

$$\left\{\underset{1}{\mathrm{R}}^{-1}(x)\right\}^{\frac{1}{m}} = v^{n\,.\,r\,.\,.}$$

$$\left\{\underset{1}{\mathrm{R}}^{-1}(x)\right\}^{\frac{1}{n}} = v^{m\,.\,r\,.\,.\,.}$$

$$\&c. = \&c.$$

which substituted make the expression rational.

Cor. 1. The more general form is this:

If $\underset{m}{\mathrm{R}}$ can be so assumed that $\underset{2}{\mathrm{R}}^{-1}\underset{m}{\mathrm{R}}$, $\underset{3}{\mathrm{R}}^{-1}\underset{m}{\mathrm{R}}$, $\underset{n}{\mathrm{R}}^{-1}\underset{m}{\mathrm{R}}$ shall be all rational; then by assuming $\underset{1}{\mathrm{R}}^{-1}(x) = \underset{m}{\mathrm{R}}(v)$ we can render rational

$$dx.\, \mathrm{R}\left\{x, \underset{1}{\mathrm{R}}^{-1}(x), \underset{2}{\mathrm{R}}^{-1}\underset{1}{\mathrm{R}}^{-1}(x), \ldots \underset{n}{\mathrm{R}}^{-1}\underset{1}{\mathrm{R}}^{-1}(x)\right\}$$

Cor. 2. We can find

$$\int dx \,.\, \mathrm{R}\left\{x, x^{\frac{1}{m}}, x^{\frac{1}{n}}, x^{\frac{1}{r}}, \ldots\right\}$$

$$\int dx \,.\, \mathrm{R}\left\{x, (\alpha x+\beta)^{\frac{1}{m}}, (\alpha x+\beta)^{\frac{1}{n}}, \ldots\right\}$$

$$\int dx \,.\, \mathrm{R}\left\{x, \left(\frac{\alpha x+\beta}{ax+b}\right)^{\frac{1}{m}}, \left(\frac{\alpha x+\beta}{ax+b}\right)^{\frac{1}{n}}, \ldots\right\}$$

$$\int dx \,.\, R\left\{x, \sqrt{\alpha^2 x^2 + \beta x + \gamma}, (\alpha x + \sqrt{\alpha^2 x^2 + \beta x + \gamma})^{\frac{1}{m}}, \ldots\ldots\right\}$$

with an indefinite number of forms too complex for convenient expression.

Cor. 3. This form may be extended to Prop. II. and other general expressions. Thus we know

$$\int dx . R\left\{x, \underset{1}{R}^{-1}(x), \underset{2}{R}^{-1}\underset{1}{R}^{-1}(x), \ldots (\underset{n}{R}^{-1} .. \underset{1}{R}^{-1}(x))^{\frac{1}{m}}, (\underset{n}{R}^{-1} \ldots \underset{1}{R}^{-1}(x))^{\frac{1}{n}}, \ldots\right\}$$

The forms given above are wholly inapplicable, when the fluxion involves expressions, such as $\underset{1}{R}^{-1}\underset{2}{R}\ \underset{3}{R}^{-1} \ldots (x)$ where the functions are alternately inverse and direct. The cases are very few, in which the difficulty can be overcome, and perhaps the following Propositions will be found to include all the instances, in which analysts have effected the reduction.

Prop. IV.

We can rationalize

$$dx \,.\, D\underset{2}{R}(x) \,.\, \left\{\underset{2}{R}(x), \underset{1}{R}^{-1}\underset{2}{R}(x)\right\}$$

Let $\underset{2}{R}(x) = v$, and it becomes

$$dv \,.\, R\left\{v, \underset{1}{R}^{-1}(v)\right\} \text{ as in Prop. 1.}$$

Cor. 1. We can generally reduce $\int dx \,.\, D\phi(x) \,.\, \underset{1}{\phi}\,\phi(x)$ to $\int dv \,.\, \underset{1}{\phi}(v)$. Thus we deduce from $\int dx \,.\, x^{rn-1} . \phi(x^n)$ the $\int dv \,.\, v^{r-1}\phi(v)$, and from $\int \frac{dx}{x} . \phi(x^n)$ the $\int \frac{dv}{v} \,.\, \phi(v)$, reductions of frequent occurrence, by which analysts have given their forms an appearance of generalization without the reality.

Cor. 2. This form may be extended to all the former Propositions.

Cor. 3. As it is very tedious and often impracticable to find x in terms of v, in order to know whether the reduction be applicable; the following process may sometimes be useful. Let the expression be

$$dx \,.\, \mathrm{D}\underset{2}{\mathrm{R}}\,(x) \,.\, \underset{1}{\mathrm{R}}^{-1}\,\underset{2}{\mathrm{R}}\,(x) .\, \mathrm{R}\,\underset{2}{\mathrm{R}}(x)$$

Then if it be divided by $dx \,.\, \mathrm{D}\underset{2}{\mathrm{R}}\,(x) \,.\, \underset{1}{\mathrm{R}}^{-1}\underset{2}{\mathrm{R}}\,\,(x)$ the quotient will be a rational function of $\underset{2}{\mathrm{R}}\,(x)$ or of the form $\frac{\underset{0}{\alpha} + \underset{1}{\alpha} \,.\, \underset{2}{\mathrm{R}}(x) + \underset{2}{\alpha} \,.\, (\underset{2}{\mathrm{R}}\,(x))^2 + \,..}{\underset{0}{a} + \underset{1}{a} \,.\, \underset{2}{\mathrm{R}}(x) + \underset{2}{a} \,.\, (\underset{2}{\mathrm{R}}\,(x))^2 + \,..}$ the coefficients being indeterminate. If the reduction be applicable, these may be found, and the substitution made at once.

Cor. 4. We may thus reduce

$$\int dx \,.\, \frac{nx^{n-1} + (n-1)\,\underset{1}{c} \,.\, x^{n-2}}{\sqrt[m]{x^n + \underset{1}{c}x^{n-1} + \ldots}} \,.\, \mathrm{R}\,\left\{x^n + \underset{1}{c}\,x^{n-1} + \ldots\right\}$$

to $\int \frac{dv}{v^{\frac{1}{m}}} .\, \mathrm{R}\,(v)$, which may be found.

Cor. 5. In $dx \,.\, x^{rn-1} \,.(\alpha x^n + \beta)^\mu \,.\, \mathrm{R}\left\{\alpha x^n + \beta\right\}$ divide by $dx \,.\, n\alpha x^{n-1} \,.\, (\alpha x^n + \beta)^\mu$, and the quotient is $\frac{1}{n\alpha} \,.\, x^{(r-1)^n} \,.\, \mathrm{R}\,(\alpha x^n + \beta)$. The latter factor is already of the required form, and by assuming

$$x(r-1)^n = \underset{0}{a} \,.\, (\alpha x^n + \beta)^{r-1} + \underset{1}{a} \,.\, (\alpha x^n + \beta)^{r-2} + \,..$$

the indeterminates may be found. In particular cases there are readier processes, but this method is universally applicable.

Prop. V.

We can rationalize

$dx \,.\, \mathrm{R}\left\{x, \underset{1}{\mathrm{R}}^{-1}\underset{n}{\mathrm{R}}(x)\right\}$ if we can determine, $\underset{1}{\mathrm{R}}^{-1}\underset{n}{\mathrm{R}}(x) = \underset{\nu}{\mathrm{R}}(x) \,.\, \underset{2}{\mathrm{R}}^{-1}(x)$, for the fluxion then becomes

$dx \,.\, \mathrm{R}\left\{x, \underset{\nu}{\mathrm{R}}(x) \,.\, \underset{2}{\mathrm{R}}^{-1}(x)\right\}$ as in Prop. I. Cor. 1.

Cor. 1. We also know

$$dx \,.\, \mathrm{R}\left\{x, \underset{2}{\mathrm{R}}^{-1}(x), \underset{1}{\mathrm{R}}^{-1}\underset{n}{\mathrm{R}}(x)\right\}$$

Cor. 2. We may thus transform

$$dx \,.\, \mathrm{R}\left\{x, \left(\frac{\alpha x + \beta}{ax + b}\right)^{\frac{1}{m}}, \sqrt[n]{(\alpha x + \beta) \,.\, (ax + b)^{n-1}}\right\}$$

into

$$dx \,.\, \mathrm{R}\left\{x, \left(\frac{\alpha x + \beta}{ax + b}\right)^{\frac{1}{m}}, (ax + b) \,.\, \left(\frac{\alpha x + \beta}{ax + b}\right)^{\frac{1}{n}}\right\}$$

a known form.

Hence we know

$$\int dx \,.\, \mathrm{R}\left\{x, \sqrt[3]{(\alpha^2 x^2 - \beta^2) \,.\, (\alpha x \pm \beta)}\right\}$$

$$\int dx \,.\, \mathrm{R}\left\{x, \sqrt[3]{(x + \alpha) \,.\, (x + \beta)^2}\right\}$$

$$\int dx \,.\, \mathrm{R}\left\{x, \sqrt{(\alpha x + \beta) \,.\, (ax + b)}\right\}$$

$$\int dx \,.\, \mathrm{R}\left\{x, \sqrt{(cx^2 + \underset{1}{c}\, x + \underset{2}{c}}\right\}$$

which last form will sometimes introduce imaginaries, that may be avoided by particular artifices.

Cor. 3. If $\underset{1}{\mathrm{R}}^{-1}\underset{m}{\mathrm{R}}(x) = \left(\underset{\mu}{\mathrm{R}}(x)\right)^m \,.\, \underset{2}{\mathrm{R}}^{-1}(x)$

$$\underset{1}{\mathrm{R}}^{-1}\underset{n}{\mathrm{R}}(x) = \left\{\underset{\nu}{\mathrm{R}}(x)\right\}^n . \underset{2}{\mathrm{R}}^{-1}(x)$$

&c. = &c.

we can determine

$$\int dx \,.\, R\left\{x, \underset{2}{R}^{-1}(x), \left\{\underset{1}{R}^{-1}\underset{m}{R}(x)\right\}^{\frac{1}{m}}, \left\{\underset{1}{R}^{-1}\underset{n}{R}(x)\right\}^{\frac{1}{n}}, \ldots\right\}$$

Cor. 4. If $\underset{1}{R}^{-1}\underset{m}{R}(x) = \underset{\mu}{R}(x)\sqrt[m]{\underset{\alpha}{R}^{-1}(x)}$; $\underset{2}{R}^{-1}\underset{n}{R}(x) = \underset{\nu}{R}(x)\,.$

$$\sqrt[n]{\underset{\alpha}{R}^{-1}(x)}\text{; \&c.} = \text{\&c.},$$

we know

$$\int dx \,.\, R\left\{x, \underset{\alpha}{R}^{-1}(x), \underset{1}{R}^{-1}\underset{m}{R}(x), \underset{2}{R}^{-1}\underset{n}{R}(x), \underset{3}{R}^{-1}\underset{r}{R}(x), \ldots\right\}$$

as in

$$dx \,.\, R\left\{x, \sqrt[m]{(\alpha x+\beta)\,.\,(ax+b)^{m-1}}, \sqrt[n]{(\alpha x+\beta)\,.\,(ax+b)^{n-1}}, \ldots\right\}$$

Cor. 5. Generally if $\underset{1}{R}(x)$ and $\underset{2}{R}(x)$ are so related, that $\underset{1}{R}\,\underset{n}{R}(x)$ can $= \underset{2}{R}\,\underset{m}{R}(x)$, $\underset{n}{R}$ and $\underset{m}{R}$ being any rational functions whatever taken at pleasure, then $dx \,.\, R\left\{x, \underset{1}{R}^{-1}\underset{2}{R}(x)\right\}$ can be rationalized by taking $x = \underset{m}{R}(v)$. It then becomes

$$dv \,.\, D\underset{m}{R}(v) \,.\, R\left\{\underset{m}{R}(v), \underset{n}{R}(v)\right\}.$$

PROP. VI.

By combining Prop. IV. and V, we can rationalize

$$dx \,.\, D\underset{m}{R}(x) \,.\, R\left\{\underset{m}{R}(x), \underset{1}{R}^{-1}\underset{n}{R}\,\underset{m}{R}(x)\right\}$$

if $\underset{1}{R}^{-1}\underset{n}{R}(v) = \underset{\nu}{R}(v) \,.\, \underset{2}{R}^{-1}(v)$; for let $\underset{m}{R}(x) = v$, and it becomes

$$dv \,.\, R\left\{v, \underset{\nu}{R}(v) \,.\, \underset{2}{R}^{-1}(v)\right\} \text{ as before.}$$

Cor. 1. If we have

$$dx \,.\, x^{m-1}(\alpha x^n + \beta)^{\frac{p}{q}} \,.\, R\left\{\alpha + \beta x^{-n}\right\}$$

Remove the multiplier x^n, as in Prop. V., and it becomes,

$$\frac{dx}{x} \cdot x^{m+\frac{np}{q}} \cdot (\alpha + \beta x^{-n})^{\frac{p}{q}} \cdot R(\alpha + \beta x^{-n})$$

which will fall under Prop. IV. If $x^{m+\frac{np}{q}}$ can be expressed by a rational function of $\alpha + \beta x^{-n}$. This will happen if $m + \frac{np}{q} = -n \cdot r$, or if $\frac{m}{n} + \frac{p}{q} = \pm r$ any integer. Hence we know

$$\int \frac{dx}{(ax^{\alpha} + bx^{\beta})^{\frac{1}{\alpha}}} \cdot R\left\{a + bx^{\beta-\alpha}\right\}$$

$$\int \frac{dx}{\sqrt[n]{\alpha + \beta x^n}}; \int \frac{dx}{\sqrt[3]{x^3 + 1}}; \int \frac{dx}{\sqrt[4]{x^4 - 1}};$$

Cor. 2. We can determine

$$\int \frac{\alpha x^{2n} - \beta}{\alpha x^{2n} + \gamma x^n + \beta} \cdot \frac{dx}{\sqrt[n]{\alpha x^{2n} + cx^n + \beta}}$$

which becomes by Prop. 5.

$$\frac{\alpha x^{n-} - \beta x^{-n-1}}{\alpha x^n + \beta x^{-n} + \gamma} \times \cdot \frac{dx}{\sqrt[n]{\alpha x^n + \beta x^{-n} + c}}$$

Now this falls under Prop. IV. For let

$$\sqrt[n]{\alpha x^n + \beta x^{-n} + c} = v$$

$$\alpha x^n + \beta x^{-n} + \gamma = v^n + \gamma - c$$

$$\therefore (\alpha x^{n-1} - \beta x^{-n-1}) \cdot dx = v^{n-1} \cdot dv$$

Whence the fluxion becomes $\frac{v^{n-2} \cdot dv}{v^n + \gamma - c}$, of which a particular case is deduced in Legendre's Elliptic Transcendents.

Prop. VII.

If we can rationalize

$$dx \cdot R\left\{x, \varphi(x), \varphi_{1}(x), \ldots\right\}$$

we also can

$$dx \,.\, \mathrm{R}\left\{x, \underset{1}{\mathrm{R}}^{-1}(x), \varphi \underset{1}{\mathrm{R}}^{-1}(x), \underset{2}{\varphi} \underset{1}{\mathrm{R}}^{-1}(x), \ldots\right\}$$

for by taking $\underset{1}{\mathrm{R}}^{-1}(x) = v$, it is reduced to the former form.

Cor. 1. If we can rationalize

$$dx \,.\, \mathrm{R}\left\{x, \underset{1}{\mathrm{R}}^{-1} \underset{n}{\mathrm{R}}(x)\right\}, \text{ we also can}$$

$$dx \,.\, \mathrm{R}\left\{x, \underset{m}{\mathrm{R}}^{-1}(x), \underset{1}{\mathrm{R}}^{-1} \underset{n}{\mathrm{R}} \underset{m}{\mathrm{R}}^{-1}(x)\right\}$$

Therefore we can find

$$\int dx \,.\, \mathrm{R}\left\{x, \underset{m}{\mathrm{R}}^{-1}(x), \sqrt{a + b \,.\, \underset{m}{\mathrm{R}}^{-1}(x) + c \,.\, \left(\underset{m}{\mathrm{R}}^{-1}(x)\right)^2}\right\}$$

$$\int dx \,.\, \mathrm{R}\left\{x, \sqrt{\alpha + \beta x + \sqrt{a + bx}}\right\} \text{ \&c.}$$

Cor. 2. Generally we can reduce

$$dx \,.\, \mathrm{R}\left\{x, \underset{0}{\mathrm{R}} \underset{1}{\mathrm{R}}^{-1} \ldots \underset{n}{\mathrm{R}} \underset{n+1}{\mathrm{R}}^{-1}(x)\right\} \text{ to}$$

$$dx \,.\, \mathrm{R}\left\{x, \underset{1}{\mathrm{R}}^{-1} \ldots \underset{n}{\mathrm{R}}(x)\right\}$$

Cor. 3. In $\int dx \,.\, \varphi(x)$, let $v = \varphi(x)$, and if it be an algebraic function, $\mathrm{R}\left\{x, v,\right\} = o$. Now take $x = \underset{1}{\mathrm{R}}(z)$ $= \dfrac{\alpha + \underset{1}{\alpha} z + \underset{2}{\alpha} z^2 + \text{\&c.}}{a + \underset{1}{a} z + \underset{2}{a} z^2 + \text{\&c.}}$; and $v = \underset{2}{\mathrm{R}}(z) = \dfrac{\beta + \underset{1}{\beta} z + \text{\&c.}}{b + \underset{1}{b} z + \text{\&c.}}$ with indeterminate coefficients.

Hence we have $\mathrm{R}\left\{\underset{1}{\mathrm{R}}(z), \underset{2}{\mathrm{R}}(z)\right\} = o$; remove fractions and make the coefficients of the powers of z vanish. This will give the indeterminates, if x and v admit common rationalities. Thence we have $dz \,.\, \mathrm{D}\underset{1}{\mathrm{R}}(z) \,.\, \underset{2}{\mathrm{R}}(z)$ rational.

Should all the artifices in the foregoing propositions fail, we must attempt to resolve the fluxion into a series of terms, such that each term may be separately rationalized. This is often possible, when the original function does not admit a rational expression, and can be effected sometimes directly, and sometimes by introducing a new variable. But it will first be necessary to reduce all irrational functions whatever to a definite form.

LEMMA.

To reduce all irrational functions to a definite form.

1. By successively multiplying numerators and denominators into the same expressions, every irrational function may at last be reduced to a series of terms, whose numerators and denominators do not contain any fraction or negative index.

Thus $\frac{\left(\alpha + \frac{\beta}{\gamma}\right)^{\mu}}{(a + bc^{-n})^{m}} = \frac{c^{nm} \cdot (\alpha\gamma + \beta)^{\mu}}{\gamma^{\mu} \cdot (ac^{n} + b)^{m}}$, and if $\alpha, \beta, \gamma, a, b, c,$ are functions involving fractions or negative indices, themselves, the reduction is continued in the same manner.

2. Now multiply both the numerators and denominators of the expressions so reduced, by such multipliers, as will render the denominators rational. This factor is the product of all the different values of the denominator, with the exception of the denominator itself. The new numerators will still consist of a series of terms not involving any fraction or negative index.

3. If $R, \underset{1}{R}, \underset{2}{R}$, &c. denote functions of the form $cx^{m} + \underset{1}{c} \cdot x^{m-1} + \ldots$, the irrational takes the form,

$$\frac{R(x)}{\underset{1}{R}(x)} + \frac{\underset{2}{R}(x)}{\underset{3}{R}(x)} \cdot \underset{1}{\varphi}(x)^{\frac{a}{b}} \cdot \underset{2}{\varphi}(x)^{\frac{c}{d}} \ldots + \frac{\underset{4}{R}(x)}{\underset{5}{R}(x)} \cdot \underset{1}{\pi}(x)^{\frac{\alpha}{\beta}} \ldots + \ldots$$

4. By reducing the fractional indices of the factors to the

common denominator (n), the whole will consist of a series of terms $\frac{\underset{2}{R}(x)}{\underset{3}{R}(x)} \cdot \sqrt[n]{\underset{1}{\phi}(x)^p \cdot \underset{2}{\phi}(x)^q \ldots}$

5. By expanding all the integer powers under the index $\frac{1}{n}$; and again reducing the indices of the sums and products, which are under it, to a common denominator n'; we shall by continuing the same operations, ultimately reduce the whole expression, to a series of terms of the form

$$\frac{\underset{2}{R}(x)}{\underset{3}{R}(x)} \sqrt[n]{S^{n'}\sqrt{S^{n''}\sqrt{\ldots S^{n''\ldots}\sqrt{\underset{m}{R}(x)}}}}$$

S denoting the sum of any number of terms such as follow it, wherein $\underset{m}{R}(x)$ may be different in each term, but always of the form $cx^m + \underset{1}{c}x^{m-1} + \ldots +$.

6. If *every* value of $\underset{m}{R}(x)$ contains a factor $(ax^r + bx^{r-1} + \ldots)^{n \cdot n' \cdot n'' \ldots}$, it may be taken entirely out of the radical; and conversely the rational coefficient may be introduced entirely under the radical.

7. When the surd is so reduced, that no rational factor can be withdrawn from the radical, it is said to be in its lowest terms; and is said to be an irrational of the 1st, 2d, or ν^{th} order, according to the number of the indices $\frac{1}{n}$, $\frac{1}{n'}$, $\frac{1}{n^{\prime \ldots \nu - 1}}$. Thus the general expression for a surd of the first order

is a series of terms, $\frac{\underset{2}{R}(x)}{\underset{3}{R}(x)} \sqrt[n]{cx^m + \underset{1}{c}x^{m-1} + \ldots}$

8. A more convenient general form for all irrationals, than

the series of terms above exhibited, may readily be found; by introducing all the rational parts entirely under the radicals; by reducing the indices of all the terms to a common denominator μ; by expanding all integer powers; and by again reducing all the products and sums contained under $\frac{1}{\mu}$, to indices with a common denominator μ'. These operations continued, will ultimately lead to the expression

$S \sqrt[\mu]{S \sqrt[\mu']{\ldots S \sqrt[\mu'\ldots]{\frac{\underset{\alpha}{R}(x)}{\underset{\beta}{R}(x)}}}}$, where $\frac{\underset{\alpha}{R}(x)}{\underset{\beta}{R}(x)}$ may be of any different values in the different sums, but always of the form

$$\frac{ax^{\alpha} + a_1 x^{\alpha-1} + \ldots}{bx^{\beta} + b_1 x^{\beta-1} + \ldots}$$

9. $\frac{\underset{\alpha}{R}(x)}{\underset{\beta}{R}(x)}$ is said to be of $\alpha - \beta$ dimensions; and if $\alpha - \beta$ be dimensions of that rational part, whose dimensions are greatest; then the dimensions of the whole irrational are $\frac{\alpha-\beta}{\mu \cdot \mu' \cdot \mu'' \ldots}$

10. The fluxion, and its dimensions in any irrational, may be found by applying this formula, $d\left\{\varphi_1 \varphi_2 \varphi_3 \ldots \varphi_n(x)\right\} = D\varphi_1\varphi_2 \ldots \varphi_n(x) \,.\, D\varphi_2\varphi_3 \ldots \varphi_n(x) \ldots D\varphi_n(x) \,.\, dx$, the D only referring to the functional characteristic immediately succeeding it.

Prop. VIII.

To divide a fluxion into expressions admitting distinct rationalities.

Let $\varphi\ (x)$ be any irrational, and $\underset{1}{\varphi}\ (x)$, $\underset{2}{\varphi}\ (x)$ &c. surds deduced as in the Lemma. Then

$$\int dx\ .\ \varphi(x) = \int \frac{dx\ .\ \text{R}\ x)}{\underset{1}{\text{R}}\ (x)} + \int \frac{dx\ .\ \underset{2}{\text{R}}\ (x)}{\underset{3}{\text{R}}\ (x)}\ .\ \underset{1}{\varphi}\ (x) + \&\text{c.}$$

where the fluent of the 1st term may always be found, and the other terms may often be rationalized by distinct substitutions, when we are unsuccessful with $\int dx.\ .\ \varphi\ (x)$. Again since in each of the terms,

$\int \frac{dx\ .\ \underset{2}{\text{R}}\ (x)}{\underset{3}{\text{R}}\ (x)}.\ \underset{1}{\varphi}\ (x)$, $\frac{\underset{2}{\text{R}}\ (x)}{\underset{3}{\text{R}}\ (x)}$ may be reduced to a series of terms of the form $\text{A}x^n$ and $\frac{\text{A}}{(x+a)^n}$; therefore the fluent depends on a series of terms $\int dx\ .\ x^n\ .\ \underset{1}{\varphi}\ (x)$, and $\int dx\ .\ (x+a)^{-n}\ .\ \underset{1}{\varphi}\ (x)$. In the latter case, the form of $\underset{1}{\varphi}\ (x)$ is not changed by substituting x for $x + a$, and ∴ the fluents of all irrationals are determinable by $\int dx\ .\ x^{\pm n}\ .\ \underset{1}{\varphi}\ (x)$.

Cor. 1. If we multiply the denominator of

$$\frac{dx\ .\ \text{R}\ (x)}{\underset{1}{\text{R}}\ (x)\ .\ \sqrt{ax^2 + bx + c} + \underset{2}{\text{R}}\ (x)\ .\ \sqrt{\alpha x^2 + \beta x + \gamma}}$$

by its rationalizing factor, the fluxion will be reduced to two terms, which admit distinct rationalities.

Cor. 2. Sometimes by the substitution of a new variable, for some function of x, the fluxion will be divided into a series of terms, each of which may be separately made rational. But unfortunately no general principle has been discovered, to which these reductions can be referred.

Cor. 3. As the fluent of each term can sometimes be found apart, when the fluent of the whole cannot be found at once; so conversely, the fluent of a series of terms may be found, when each separate term surpasses the powers of analysis. Thus we know

$$\int \frac{d\varphi(x) + d\underset{1}{\varphi}(x) + \dots}{\varphi(x) + \underset{1}{\varphi}(x) + \dots}$$

But we do not know

$$\int \frac{d\varphi(x)}{\varphi(x) + \underset{1}{\varphi}(x) + \dots} + \int \frac{d\underset{1}{\varphi}(x)}{\varphi(x) + \underset{1}{\varphi}(x) + \dots} + \dots$$

Again, let $\varphi(x)$ be such a function of x,

that $\varphi^2(x) = \frac{x}{e}$; let $\varphi(x) = \underset{1}{x}$;

$$\text{Then} \int dx \,.\, \varphi(x) = x \,.\, \varphi(x) - \int d\varphi(x) \,.\, x$$
$$= x \,.\, \varphi(x) - \int d\varphi(x) \,.\, \varphi^2(x) \,.\, e$$
$$= x \,.\, \underset{1}{x} \quad - e \int d\underset{1}{x} \,.\, \varphi(\underset{1}{x})$$

$$\therefore \int dx \,.\, \varphi(x) + e \int d\underset{1}{x} \,.\, \varphi(\underset{1}{x}) = x \,.\, \underset{1}{x}$$

Which theorem admits farther extension, and may be applied to elliptic arches.

Should the above processes for rendering the fluxion rational fail us, we must attempt the fluxion at once in its irrational state, for which purpose I shall add a few miscellaneous observations.

1. If $\varphi(x)$, $\underset{1}{\varphi}(x)$ be any algebraic functions,

then $d\left\{\varphi(x) + \log. \underset{1}{\varphi}(x)\right\} = d\varphi(x) + \frac{d\underset{1}{\varphi}(x)}{\underset{1}{\varphi}(x)}$ is an algebraic expression. Whenever, therefore, we meet with an algebraic fluxion, we may legitimately try $\varphi(x) + \log. \underset{1}{\varphi}(x)$, as a form to which the fluent may possibly belong.*

2. It presents three cases: 1st. where the fluent is wholly algebraic, for which we assume some expression so general, that its fluxion will include the given fluxion, if it admit an algebraic fluent; or we find the fluent implicitly by an equation: 2dly. where the fluent is mixed, when we attempt to separate the algebraic part: 3dly. where the fluent is purely logarithmic, when we assume, as in the first case, some expression with indeterminate constants, sufficiently general to include the given fluxion.

3. In assuming for an algebraic fluxion, it must be observed, that the fluent will always be a surd of the same order as the fluxion. On this principle WARING gives some assumptions for surds of the second order, but nothing has been attempted generally for surds of all orders, for want of some definite form which should include them all. In irrationals of the first order, the fluxion may always be reduced to series of terms, such as

$$dx \cdot \left(x + \underset{1}{a}\right)^{\alpha} \cdot \left(x + \underset{2}{a}\right)^{\beta} \ldots \left(x + \underset{n}{a}\right)$$

where the factors are all different, and where the indices are positive, negative, fractions, integers, or unity. Then let $\underset{n-1}{R}(x)$ be any expression $cx^{n-1} + \underset{1}{c}x^{n-2} + \ldots$ with indeterminate coefficients. Assume for the fluent

$$\frac{\left(x + \underset{1}{a}\right)^{\alpha+1} \cdot \left(x + \underset{2}{a}\right)^{\beta+1} \ldots \left(x + \underset{n}{a}\right)^{\nu+1}}{\underset{n-1}{R}(x)}$$

* It is obvious, that the fluent of an algebraic fluxion cannot be of the form $\varphi(x) + \underset{1}{\varphi}(x) \cdot \log. \underset{2}{\varphi}(x)$, for its fluxion $d\varphi(x) + d\underset{1}{\varphi}(x) \cdot \log. \underset{2}{\varphi}(x) + \underset{1}{\varphi}x \cdot \frac{d\underset{2}{\varphi}(x)}{\underset{2}{\varphi}(x)}$ is a transcendent.

Its fluxion will be

$$\frac{DR_{n-1}(x) \cdot \left(x+a_1\right) \cdot \left(x+a_2\right) \cdots \left(x+a_n\right)}{\left\{R_{n-1}(x)\right\}^2}$$

$$-\frac{R_{n-1}(x) \cdot \left\{(\alpha+1) \cdot \left(x+a_2\right) \cdots \left(x+a_n\right)+(\beta+1) \cdot \left(x+a_1\right) \cdots \left(x+a_n\right)+ \cdot\right\}}{\left\{R_{n-1}(x)\right\}^2}$$

multiplied by $dx \cdot (x+a_1)^{\alpha} \ldots (x+a_n)^{\nu}$ the original fluxion.

Now that the two expressions may be equal, the coefficient found above must be $=1$, or we must have

$$DR_{n-1}(x) \cdot Q - R_{n-1}(x) \cdot Q' = \left\{R_{n-1}(x)\right\}^2$$

where Q and Q′ are the expressions in the coefficient involving a_1, a_2, a_n. By equating the terms in this equation, the indeterminates c_1, c_2, c, &c. may be found; but the reduction will often be impossible, as there are more equations to be satisfied than there are indeterminates.

4. If any index $\alpha, \beta, \gamma = -1$, the expression fails, and there is no algebraic fluent; also WARING says, that if the dimensions of a fluxional coefficient be $= -1$, the fluent cannot be algebraic.

5. If $\phi(x)$ be an irrational function, let $z = \int dx \cdot \phi(x) = \int dx \cdot v$; then since $R\left\{x, v\right\} = o$, there are cases, where we can determine, $R_1\left\{x, z\right\} = o$.*

6. If $\phi(x)$, $\pi(x)$ be irrational functions of x, we have

$$\int dx \cdot \phi(x) = \int dx \cdot \left\{\phi(x) + \pi(x)\right\} - \int dx \cdot \pi(x)$$

Now let $\pi(x)$ be so assumed, that

* See Phil. Trans. 1764.—EMERSON's Fluxions.

$\int dx \,.\, \left\{\varphi(x) + \pi(x)\right\} = \underset{1}{\varphi}(x)$, and we have

$$\int dx \,.\, \varphi(x) = \underset{1}{\varphi}(x) - \int dx \,.\, \pi(x).$$

If therefore $\pi(x)$ be a simpler expression than $\varphi(x)$, the fluxion will be reduced to a simpler form. In order to find $\pi(x)$; $\underset{1}{\varphi}(x)$ is assumed with indeterminate coefficients, so that its fluxion may be of the same form as $dx \,.\, \varphi(x)$. Now equate the similar terms in the two expressions, and the indeterminates may be found. But as there may be more equations than indeterminates, we add $\pi(x)$ a function of the same form, and containing indeterminates of sufficient number, to satisfy all the deficient equations. We shall thus have $D\underset{1}{\varphi}(x) = \varphi(x) + \pi(x)$ and $\therefore \underset{1}{\varphi}(x) = \int dx. \left\{\varphi(x) + \pi(x)\right\}$ by which the difficulty may be reduced to finding $\int dx \,.\, \pi(x)$. Reductions of particular kinds were discovered by SIMPSON and others, but this is universally applicable.

7. It may be of advantage to reduce the index of the variable under the radical, which may sometimes be effected. In $dx \,.\, \left\{x^{m+n} + 1\right\}^{\frac{1}{n}}$, assume $x^{m+n} + 1 = v^n$;

Then we have

$$dx \,.\, \left\{x^{m+n} + 1\right\}^{\frac{1}{n}} = \frac{n}{m+n} \,.\, dv \,.\, v^n \,.\, (v^n - 1)^{\frac{1}{m+n} - 1}$$

And in the same manner surds of one order may be transformed into another.

8. If the fluent be wholly logarithmic, we may assume for irrationals of the first order

$$\log. \left\{\underset{0}{R}(x) + \underset{1}{R}(x).(x+a)^{\alpha}.(x+b)^{\beta}\ldots + \underset{2}{R}(x).(x+c)^{\gamma}\ldots + \right\}$$

9. I shall conclude by observing that the fluxion may always be made rational, if the fluent be wholly algebraic, or wholly logarithmic. Thus, if $\varphi\,(x)$ be any algebraic function, take $x = \varphi^{-1}\,\mathrm{R}\,(v)$,

Then $d\varphi\,(x) \quad = d\left(\varphi\varphi^{-1}\,\mathrm{R}\,(v)\right) = d\,\mathrm{R}\,(v)$

and $d\left(\log.\,\varphi(x)\right) = d\left(\log.\,\varphi\varphi^{-1}\,\mathrm{R}\,(v)\right) = \frac{d\,\mathrm{R}\,(v)}{\mathrm{R}\,(v)}$

are both rational. If the fluent be of the mixed form $\varphi\,(x) + \log.\,\varphi_1\,(x)$, its fluxion may be made rational, if R, R_1, can be so assumed that $\varphi^{-1}\,\mathrm{R}\,(x) = \varphi_1^{-1}\,\mathrm{R}_1\,(x)$; and it may always be effected by introducing two new variables.

First let $x = \varphi^{-1}\,\mathrm{R}\,(v)$, and the fluxion becomes

$d\,\mathrm{R}\,(v) + \frac{d\left(\varphi_1\varphi^{-1}\mathrm{R}\,(v)\right)}{\varphi_1\varphi^{-1}\mathrm{R}\,(v)}$; now let $v = \mathrm{R}^{-1}\,\varphi\varphi_1^{-1}\,\mathrm{R}_1\,(z)$, an

we get $d\,\mathrm{R}\,(v) + \frac{d\,\mathrm{R}_1\,(z)}{\mathrm{R}_1\,(z)}$ which is wholly rational.

EDWARD FFRENCH BROMHEAD.

Note.—As modern analysts have in general confounded the fluxions, either with the increments or the derived functions, it may not be superfluous to state precisely, what is meant by the symbols d and D.

If it be possible, (which must be shown in each particular case) to expand $\varphi\,(x+v)$ in the form $\varphi\,(x) + \varphi_1\,(x)\,.\quad + \varphi_2\,(x)\,.\,v^2 + \ldots$; then $\varphi_1\,(x)$ is called the derived function of $\varphi\,(x)$, and its relation to $\varphi\,(x)$ is thus expressed, $\varphi_1\,(x) = \mathrm{D}\varphi\,(x)$. Hence, if x be considered a function of itself, we have $(x + v) = (x) + \mathrm{D}\,(x)\,.\,v$, and $\therefore$ $\mathrm{D}\,(x) = 1$.

Now to avoid a constant reference to the variable x, of which other variables are considered as functions, we introduce fluxions. If $y, z, w, \ldots$ are functions of the same variable, then $dy, dz, dw, \ldots$ are expressions ***proportional*** to the derived functions of $y, z, w, \ldots$ whatever may be the variable of which they are common functions. Hence $\frac{dy}{dz} = \frac{Dy}{Dz}$; and if y be a function of x, or $= \varphi(x)$, then $\frac{dy}{dx} = \frac{D\varphi(x)}{Dx} = D\varphi(x)$ and $\therefore\ dy = dx \,.\, D\varphi(x)$.

Moreover, since the derived functions are in the limiting ratio of the increments, so also are the fluxions. From this consideration we can in the applications of analysis, ***practically*** determine the ratio of the fluxions, when the derived functions are unknown.

ERRATA.

Page 72, line 20, for ***parts***, read ***part***.
—— 73, line 3, for *between*, read *below*.
—— 98, line 4 from bottom, dele the comma after A.
—— 101, line 6 from bottom, dele BH.
—— 102, line 4, for ***axes***, read ***axis***.
—— 164, line 11, dele the comma between m and n.
—— 174, line 7, for ***consisted of***, read ***consisted in***.
—— ——, line last, for m, n, read m, m.
—— 191, line 13, for $\varphi\overset{-1}{\varphi}x$, read $\overset{-1}{\varphi}\varphi x$.
—— 213, line 14, for $\psi^{p}\,\psi(x, y)$, read $\varphi^{p}\,\psi(x, y)$.
—— 214, line 10, dele "***in an infinite number of ways***".
—— 224, line 22, for $f(a)$, read $f(x)$.
—— 226, line 24, for $= x$, read $= z$.
—— 232, line 16, ***in the denominator***, for $1-$, read $1+$.
—— ——, line 18, *ditto*, *ditto*, for $1-$, ***read*** $1\pm$.
—— 251, line 9, for $\frac{d\psi\, x, \frac{1}{y})}{dx}$ read $\frac{d\psi\,(x, \frac{1}{y})}{dx}$
—— ——, line 11, for ***d in both numerator***, read d^2.
—— —— line 13, for $\left(\frac{x}{y}\right)$ read $x\,\varphi\left(\frac{x}{y}\right)$.